XI

TOWARDS A NON-LINEAR SCHWARZ'S LIST

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Dedicated to Nigel Hitchin for his 60th birthday

11.1 Introduction

The main theme of this chapter is 'icosahedral' solutions of (ordinary) differential equations, a topic that seems suitable for a 60th birthday conference. We will however try to go beyond the icosahedron, to see what comes next, and consider various symmetry groups each of which could be thought of as the next in a sequence, following the icosahedral group.

To fix ideas let us give a classical example. Recall the icosahedral rotation group of order 60:

$$A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5) \cong \Delta_{235} \cong \langle\ a, b, c \mid a^2 = b^3 = c^5 = abc = 1\ \rangle.$$

This is described via three generators a, b, and c whose product is the identity, and so it is natural to look for ordinary differential equations (ODEs) on the three-punctured sphere $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ with monodromy group A_5. Now A_5 is a three-dimensional rotation group so naturally lives in $\mathrm{SO}_3(\mathbb{R})$ which is a subgroup of $\mathrm{SO}_3(\mathbb{C})$ which is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$. Thus we are led to search for connections

$$\nabla = d - \left(\frac{A_1}{z} + \frac{A_2}{z-1}\right) dz, \quad A_i \in \mathfrak{sl}_2(\mathbb{C}) \tag{11.1}$$

on rank 2 holomorphic vector bundles over the three-punctured sphere with *projective* monodromy group equal to A_5.

Such connections are essentially the same as Gauss hypergeometric equations, and H. Schwarz (1873) classified all such equations having finite monodromy groups. The list he produced has 15 rows, 1 for the family of dihedral groups, 2 rows for each of the tetrahedral and octahedral groups, and 10 rows for the icosahedral group (see Table 11.1).

Note added in proof: Lisovyy and Tykhyy have recently announced (arXiv:0809.4873) that the 'Non-linear Schwarz's list' constructed here is in fact complete.

Table 11.1. Schwarz's list:

	No.	λ''	μ''	ν''	$\frac{\text{Inhalt}}{\pi}$	Polyeder
aa*	I	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regelmässige Doppelpyramide
abb	II	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}=A$	Tetraeder
bbb	III	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}=2A$	
abg	IV	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12}=B$	Würfel und Oktaeder
bgg	V	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}=2B$	
abc	VI	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{30}=C$	Dodekaeder und Ikosaeder
bbd	VII	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{15}=2C$	
bcc	VIII	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{15}=2C$	
acd	IX	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10}=3C$	
bcd	X	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{15}=4C$	
ddd	XI	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}=6C$	
bbc	XII	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5}=6C$	
ccc	XIII	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}=6C$	
abd	XIV	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{7}{30}=7C$	
bdd	XV	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}=10C$	

Source: From Schwarz (1873).

A key point here is that the Gauss hypergeometric equation is *rigid* so the full monodromy representation (of the fundamental group of the three-punctured sphere into $\mathrm{PSL}_2(\mathbb{C})$) is determined by the conjugacy classes of the monodromy around each of the punctures. Thus in Schwarz's list it is sufficient to list these local monodromy conjugacy classes in order to specify the possible monodromy representations (and from this it is easy to find a hypergeometric equation with given monodromy). To ease recognition, to the left of the table we have listed the triples of conjugacy classes which occur, labelling the four non-trivial conjugacy classes of A_5 by a, b, c, and d, representing rotations by $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$, and $\frac{2}{5}$ of a turn, respectively. (In the octahedral case one may also have rotations by a quarter of a turn, which we label by g.)

11.1.1 *Naive generalizations*

Our basic aim is to discuss three naive generalizations of Schwarz's list, as follows. The first two arise simply by looking for *non-rigid* connections that are natural generalizations of the hypergeometric connections considered above, obtained by adding an extra singularity – the two cases are generalizations of two ways one may view the hypergeometric equation as a connection. First of all we can simply add another pole at some point t:

$$(\mathbf{A}) \qquad \nabla = d - \left(\frac{A_1}{z} + \frac{A_2}{z-t} + \frac{A_3}{z-1}\right) dz, \quad A_i \in \mathfrak{sl}_2(\mathbb{C})$$

and keep the coefficients in $\mathfrak{sl}_2(\mathbb{C})$.

Secondly, we recall that the connection one obtains immediately upon choosing a cyclic vector for the hypergeometric equation is as in (11.1) but with A_1, A_2 both rank 1 matrices (in $\mathfrak{gl}_2(\mathbb{C})$). Then the monodromy group will be a complex reflection group (generated by two two-dimensional complex reflections[1]) and the natural generalization is then to consider connections of the form

$$(\mathbf{B}) \qquad \nabla = d - \left(\frac{B_1}{z} + \frac{B_2}{z-t} + \frac{B_3}{z-1}\right) dz, \quad B_i \in \mathfrak{gl}_3(\mathbb{C})$$

with each B_i having rank 1, so the monodromy group will be generated by three three-dimensional complex reflections. This is a very natural condition as we will see.

Questions A, B: Find the analogue of Schwarz's list for connections (**A**) or (**B**).

These questions can now be answered and lead to two 'non-rigid Schwarz's lists', that is, to classifications of possible monodromy representations with finite image (up to equivalence) and the construction of connections realizing such representations. We should emphasize that the main focus has been the construction of such connections with given monodromy representation for any value of t (which is a tricky business in this non-rigid case), rather than just the classification.

Example (of type (B)). The full symmetry group of the icosahedron is the icosahedral reflection group of order 120:

$$H = H_3 \cong \left\langle r_1, r_2, r_3 \;\middle|\; r_i^2 = 1, (r_1 r_2)^2 = (r_2 r_3)^3 = (r_3 r_1)^5 = 1 \right\rangle$$
$$\subset O_3(\mathbb{R}) \subset \mathrm{GL}_3(\mathbb{C}).$$

This is generated by three reflections (whose product is *not* the identity) and so it is natural to look for connections on rank 3 bundles over a four-punctured sphere with monodromy H (generated by three reflections about three of the punctures – that is, connections of the form (**B**) with each of the three residues B_i having trace $\frac{1}{2}$ so the corresponding reflections are of order 2). There turn out to be three inequivalent triples of generating reflections of H, two of which are related by an outer automorphism. The problem is to write down connections

[1] That is, arbitrary automorphisms of the form 'one plus rank 1', not necessarily of order 2 or orthogonal.

of the desired form for any value t of the final pole position. One triple of generating reflections is intimately related to K. Saito's flat structure (1993) for H (or icosahedral Frobenius manifold) and appears in Dubrovin's article (1995, appendix E). The other two triples were dealt with around 1997: see Dubrovin and Mazzocco (2000); one is similar to the first case (since related to it by an outer automorphism) but the final triple turned out to be much trickier, and writing out the family of connections in this case involved a specific elliptic curve which took about 10 pages of 40 digits integers to write down (see the preprint version of *op. cit.* on the mathematics arxiv). We will eventually see below that this elliptic solution is in fact equivalent to a solution with a simple parametrization, agreeing with Hitchin's philosophy that 'nice problems should have nice solutions'.

Remark. Before moving on to the third generalization let us add some other historical comments. The 'non-naive' generalizations of the Gauss hypergeometric equation are the equations satisfied by the ${}_nF_{n-1}$ hypergeometric functions (the Gauss case being that of $n = 2$). The corresponding Schwarz's list appears in Beukers and Heckman (1989). In terms of connections this amounts to considering connections (11.1) on rank n vector bundles, still with three singularities on $\mathbb{P}^1$, but with A_1 of rank $n - 1$ and A_2 of rank 1; *these connections are still rigid*.

Some work in the non-rigid case has been done (besides that we will recall below) by considering generalizations of the hypergeometric equation as an equation (rather than as a connection); for example, the algebraic solutions of the Lamé equation were studied in Beukers and van der Waall (2004) (Lamé equations are basically the second order Fuchsian equations with four singular points on $\mathbb{P}^1$ such that three of the local monodromies are of order 2). In general connections of type (**A**) with such monodromy representations will not come from a Lamé equation (since upon choosing a cyclic vector the corresponding equations will in general have additional apparent singularities; this can also be seen by counting dimensions). Indeed it turns out (*op. cit.*) that Lamé equations only have finite monodromy for special configurations of the four poles.

11.1.2 *Non-linear analogue: the Painlevé VI equation*

One reason hypergeometric equations are interesting is that they provide the simplest explicit examples of *Gauss–Manin connections*. Indeed this is one reason Gauss was interested in them: he observed that the periods of a family of elliptic curves satisfy a (Gauss) hypergeometric equation. (The modern interpretation of this is as the explicit form of the natural flat connection on the vector bundle of first cohomologies over the base of the family of elliptic curves, written with respect to the basis given by the holomorphic one-forms – and their derivatives – on the fibres.) Nowadays there is still much interest in such linear differential equations 'coming from geometry'.

Thus the non-linear analogue of the Gauss hypergeometric equation should be the explicit form of the simplest *nonabelian* Gauss–Manin connection (i.e. the explicit form of the natural connection on the bundle of first nonabelian cohomologies of some family of varieties). The simplest interesting case corresponds to taking the universal family of four-punctured spheres and taking cohomology with coefficients in $\mathrm{SL}_2(\mathbb{C})$ (one needs a non-trivial family of varieties with nonabelian fundamental groups). This leads to the Painlevé VI equation ($\mathrm{P_{VI}}$), which is a second-order non-linear differential equation whose solutions, like those of the hypergeometric equation, branch only at $0, 1, \infty \in \mathbb{P}^1$. In particular we may study the (non-linear) monodromy of solutions of $\mathrm{P_{VI}}$, by examining how solutions vary upon analytic continuation along paths in the three-punctured sphere.

Thus, since Schwarz lists fundamental solutions of hypergeometric equations having finite monodromy, our main question is to construct the analogue of Schwarz's list for $\mathrm{P_{VI}}$:

Question C: What are the solutions of Painlevé VI having finite monodromy?

This question is still open; there is as yet no full classification – the main effort (at least of the present author) has been towards finding and constructing interesting solutions. So far all known finite-branching solutions are actually algebraic (cf. Iwasaki 2008). Currently we are at the reasonably happy state of affairs that all such solutions known to exist have actually been constructed. In what follows I will explain various aspects of the problem, and in particular show how the non-rigid lists (**A**) and (**B**) map to the list of (**C**). Some key points, demonstrating the richness and variety of solutions, are

- There are algebraic solutions of $\mathrm{P_{VI}}$ not related to finite subgroups of the coefficient group $\mathrm{SL}_2(\mathbb{C})$.
- There are 'generic' solutions of $\mathrm{P_{VI}}$ with finite monodromy; that is, not lying on any of the reflection hyperplanes of the affine F_4 Weyl group of symmetries of $\mathrm{P_{VI}}$.
- There are entries on the list of (**C**) which do not come from either (**A**) or (**B**).

In particular we will see $\mathrm{P_{VI}}$ solutions related to the groups A_6, $\mathrm{PSL}_2(\mathbb{F}_7)$ and Δ_{237}.

11.2 What is Painlevé VI?

There are various viewpoints, and simply giving the explicit equation is perhaps the least helpful introduction to it. In brief, Painlevé VI is

- The explicit form of the simplest nonabelian Gauss–Manin connection
- The equation controlling the 'isomonodromic deformations' of certain logarithmic connections/Fuchsian systems on $\mathbb{P}^1$
- The most general second-order ODE with the so-called 'Painlevé property'

- A certain dimensional reduction of the anti-self-dual Yang–Mills equations (see e.g. Mason and Woodhouse (1996)),
- An equation related to certain elliptic integrals with moving endpoints (cf. Fuchs (1905) and Manin (1998))
- The second-order ODE for a complex function $y(t)$

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{(t-1)}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right)$$

  where $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ are constants

The Painlevé property means that any local solution $y(t)$ defined in a disc in the three-punctured sphere $\mathbb{P}^1\setminus\{0,1,\infty\}$ extends to a meromorphic function on the universal cover of $\mathbb{P}^1\setminus\{0,1,\infty\}$. It is this property that enables us to speak of the monodromy of $\mathrm{P_{VI}}$ solutions. Concerning solutions there is a basic trichotomy (see Watanabe (1998)):

$$\text{A solution of } \mathrm{P_{VI}} \text{ is either } \begin{cases} \text{a `new' transcendental function, or} \\ \text{a solution of a first-order Riccati equation, or} \\ \text{an algebraic function.} \end{cases}$$

In particular if one is interested in constructing new explicit solutions of $\mathrm{P_{VI}}$ then, since the Riccati solutions are all well understood, the algebraic solutions are the first place to look.

The standard approach to $\mathrm{P_{VI}}$ is as isomonodromic deformations of rank 2 logarithmic connections with four poles on $\mathbb{P}^1$, as the poles move (generic such connections are of the form (**A**), and then t parametrizes the possible pole configurations). In particular one can see the four constants in $\mathrm{P_{VI}}$ directly in terms of the eigenvalues of the residues of the connection: if we set θ_i to be the difference of the eigenvalues (in some order) of the residue A_i ($i = 1,2,3,4$, where $A_4 = -\sum_1^3 A_i$ is the residue at infinity) then the relation to the constants is

$$\alpha = (\theta_4-1)^2/2, \quad \beta = -\theta_1^2/2, \quad \gamma = \theta_3^2/2, \quad \text{and } \delta = (1-\theta_2^2)/2.$$

Before going into more detail let us also mention one further property of $\mathrm{P_{VI}}$: it admits a group of symmetries isomorphic to the affine Weyl group of type F_4 (see Okamoto (1987) or the exposition in Boalch (2006)). Indeed treating $\theta = (\theta_1,\dots,\theta_4)\in\mathbb{C}^4$ as the set of parameters for $\mathrm{P_{VI}}$ is useful since the affine F_4 Weyl group of symmetries acts in the standard way on this $\mathbb{C}^4$. (We will see below that these four parameters may also be interpreted as coordinates on the moduli space of cubic surfaces.)

11.2.1 *Conceptual approach to Painlevé VI*

Consider the universal family of smooth four-punctured rational curves with labelled punctures. Write $B := \mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$ for the base, $\mathcal{F}$ for the standard fibre, and $\mathcal{C}$ for the total space:

$$\begin{array}{ccc} \mathcal{C} & \longleftarrow & \mathcal{F} \cong \mathbb{P}^1 \setminus 4 \text{ points} \\ \big\downarrow & & \\ B & & \end{array}$$

Now replace each fibre $\mathcal{F}$ by $\mathrm{H}^1(\mathcal{F}, G)$ where $G = \mathrm{SL}_2(\mathbb{C})$. Here we will use two viewpoints/realizations of this nonabelian cohomology set H^1:

1. Betti: Moduli of fundamental group representations

$$\mathrm{H}^1(\mathcal{F}, G) \cong \mathrm{Hom}(\pi_1(\mathcal{F}), G)/G$$

2. De Rham: Moduli of connections on holomorphic vector bundles over $\mathcal{F}$

These two viewpoints are related by the Riemann–Hilbert correspondence (the nonabelian De Rham functor), taking connections to their monodromy representations. The point is that algebraically these realizations of H^1 are very different and the Riemann–Hilbert map is transcendental (things written in algebraic coordinates on one side will look a lot more complicated from the other side).

Thus we get two non-linear fibrations over the base B, with fibres the De Rham or Betti realizations of $\mathrm{H}^1(\mathcal{F}, G)$, respectively:

$$\begin{array}{ccc} \mathcal{M}_{\text{De Rham}} & \xrightarrow{\text{Riemann–Hilbert}} & \mathcal{M}_{\text{Betti}} \\ \big\downarrow & & \big\downarrow \\ B & = & B \end{array}$$

As in the case with abelian coefficients one still gets a natural connection on these cohomology bundles. The surprising fact is that it is algebraic on *both* sides (approximating the De Rham side in terms of logarithmic connections to give it an algebraic structure Nitsure 1993). Thus when written explicitly we will get *non-linear algebraic* differential equations 'coming from geometry'. (See Simpson 1994, section 8 for more on these connections in the case of families of projective varieties.)

The two standard descriptions of the abelian Gauss–Manin connection generalize to descriptions of this non-linear connection. In the Betti picture we may identify two nearby fibres of $\mathcal{M}_{\text{Betti}}$ simply by keeping the monodromy representations (points of the fibres) constant: moving around in B amounts to deforming the configuration of four points in $\mathbb{P}^1$ and it is easy to see how to identify the fundamental groups of the four-punctured spheres as the punctures are deformed – use the same generating loops. This 'isomonodromic' description, preserving the monodromy representation, is the nonabelian analogue of keeping the periods of one-forms constant.

On the De Rham side the non-linear connection can be described in terms of extending a connection on a vector bundle over a fibre $\mathcal{F}$, to a flat connection on a vector bundle over a family of fibres and then restricting to another fibre, much as the abelian case is described in terms of closed one-forms (linear connections replacing one-forms and flatness replacing the notion of closedness).

Each of these descriptions has a use: the De Rham viewpoint lends itself to giving an explicit form of the non-linear connection (essentially amounting to the condition for the flatness of the connection over the family of fibres). The Betti viewpoint is more global and allows us to study the monodromy of the non-linear connection, as an explicit action on fibres of $\mathcal{M}_{\mathrm{Betti}}$.

11.2.2 *Explicit non-linear equations*

The De Rham bundle $\mathcal{M}_{\mathrm{De\,Rham}}$ is well approximated by the space of logarithmic connections with four poles on the trivial rank 2 holomorphic bundle (with trivial determinant) over $\mathbb{P}^1$. Call the space of such connections $\mathcal{M}^*$ and observe it parametrizes connections of the form (**A**), and that these are determined by the value of $t \in B$ and the residues:

$$\mathcal{M}^* \cong B \times \left\{ (A_1, \ldots, A_4) \mid A_i \in \mathfrak{g}, \sum A_i = 0 \right\} /G.$$

Here $G = \mathrm{SL}_2(\mathbb{C})$ does not act on B and acts by diagonal conjugation on the residues A_i. In general this quotient will not be well behaved, but it has a natural Poisson structure and the generic symplectic leaves will be smooth complex symplectic surfaces. Clearly $\mathcal{M}^*$ is trivial as a bundle over B (projecting onto the configuration of poles), but the nonabelian Gauss–Manin connection is different to this trivial connection and was computed about 100 years ago by Schlesinger (essentially in the way stated above it seems). The non-linear connection is given by *Schlesinger's equations*, which in the case at hand are

$$\frac{dA_1}{dt} = \frac{[A_2, A_1]}{t}, \qquad \frac{dA_3}{dt} = \frac{[A_2, A_3]}{t-1}$$

together with a third equation for dA_2/dt easily deduced from the fact that A_4 remains constant. If the residues of the connection satisfy these equations then the corresponding monodromy representation remains constant as t varies. (They are easily derived from the vanishing of the curvature of the 'full' connection $d - \left(A_1 \frac{dz}{z} + A_2 \frac{dz-dt}{z-t} + A_3 \frac{dz}{z-1}\right)$.)

To get from here to P_{VI} one chooses specific functions x, y on $\mathcal{M}^*$ which restrict to coordinates on each generic symplectic leaf and writes down the connection in these (carefully chosen) coordinates (see Boalch 2005, pp.199–200 for a discussion of the formulae, which are from Jimbo and Miwa 1981). This leads to two coupled non-linear first-order equations, and eliminating x leads to the second-order Painlevé VI equation for $y(t)$. It was first written down in full generality by R. Fuchs (1905) (whose father L. Fuchs was also the father of 'Fuchsian equations').

11.2.3 *Monodromy of Painlevé VI*

Since the Betti and De Rham realizations are analytically isomorphic, we see the monodromy of solutions to P_{VI} thus corresponds to the monodromy of the connection on $\mathcal{M}_{\mathrm{Betti}}$. This amounts to an *action* of the fundamental group of the base B on a fibre, and this action can be described explicitly.

Let $\mathcal{M}_t = \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0,t,1,\infty\}), G)/G$ be the fibre of $\mathcal{M}_{\mathrm{Betti}}$ at some fixed point $t \in B$. The key point is that $\pi_1(B) \cong \mathcal{F}_2$ (the free nonabelian group on two generators) may be identified with the pure mapping class group of the four punctured sphere $\mathbb{P}^1 \setminus \{0,t,1,\infty\}$. As such it has a natural action on $\mathcal{M}_t$ (by pushing forward loops generating the fundamental group), and this action is the desired monodromy action.

Explicitly, upon choosing appropriate generating loops of $\pi_1(\mathbb{P}^1 \setminus \{0,t,1,\infty\})$ we see $\mathcal{M}_t$ may be described directly in terms of monodromy matrices:

$$\mathcal{M}_t \cong \left\{ (M_1,\ldots,M_4) \;\middle|\; M_i \in G,\ M_4 \cdots M_1 = 1 \right\} /G$$

which in turn is simply the quotient G^3/G of three copies of G by diagonal conjugation by $G = \mathrm{SL}_2(\mathbb{C})$. In fact this quotient has been studied classically: the ring of G invariant functions on G^3 has seven generators and one relation, embedding the affine quotient variety as a hypersurface in $\mathbb{C}^7$. The particular equation for this hypersurface appears on p. 366 of Fricke and Klein (1897). The Painlevé VI parameters essentially specify the conjugacy classes of the four monodromies M_i, and serve here to fibre the six-dimensional hypersurface G^3/G into a four-parameter family of surfaces. Looking at the explicit equation shows they are affine *cubic* surfaces. In turn Iwasaki (2002) has recently pointed out that this family of cubics may be quite simply related to the explicit family of Cayley (1849) and so contains the generic cubic surface.

The desired action of the free group $\mathcal{F}_2$ on the Betti spaces is given by the squares of the following 'Hurwitz' action:

$$\omega_1(M_1,M_2,M_3) = \left(M_2, M_2M_1M_2^{-1}, M_3\right)$$

$$\omega_2(M_1,M_2,M_3) = \left(M_1, M_3, M_3M_2M_3^{-1}\right).$$

More explicitly if we consider simple positive loops l_1, l_2 in B based at $\frac{1}{2}$ encircling $0, 1$, respectively, then the monodromy of the connection on $\mathcal{M}_{\mathrm{Betti}}$ around l_i is given by ω_i^2 (with respect to certain generators of $\pi_1\left(\mathbb{P}^1 \setminus \left\{0, \frac{1}{2}, 1, \infty\right\}\right)$. In turn it is possible to write this action directly as an action on the ring of invariant function on G^3.

11.3 Algebraic solutions from finite subgroups of $\mathbf{SL_2(C)}$

11.3.1 *What exactly is an algebraic solution?*

The obvious definition is simply an algebraic function $y(t)$ which solves $\mathrm{P_{VI}}$ for some value of the four parameters. Thus it will be specified by some polynomial equation

$$F(y,t)=0$$

and a four-tuple θ of parameters. In practice however such polynomials F can be quite unwieldy and are difficult to transform under the affine Weyl symmetry group, making it difficult to see if in fact two solutions are equivalent. This leads to our preferred definition.

Definition. *An algebraic solution of* $\mathrm{P_{VI}}$ *is a compact, possibly singular, algebraic curve* Π *together with two rational functions* $y,t:\Pi\to\mathbb{P}^1$ *:*

$$\begin{tikzcd}
\Pi \arrow[r, "y"] \arrow[d, "t"'] & \mathbb{P}^1 \\
\mathbb{P}^1 &
\end{tikzcd}$$

such that

- *t is a Belyi map (i.e. its branch locus is a subset of $\{0,1,\infty\}$).*
- *y, when viewed as a function of t away from the ramification points of t, solves* $\mathrm{P_{VI}}$ *for some value of the four parameters.*

In principle it is straightforward to go between the two definitions, but in practice it is useful to look for a good model of Π (and the model given by the closure of the zero locus of the polynomial F is usually a bad choice).

11.3.2 $(\mathbf{A})\mapsto(\mathbf{C})$

Suppose we have a linear connection $(\mathbf{A})$ with finite monodromy. Its monodromy representation will be specified by a triple $(M_1,M_2,M_3)\in G^3$ generating a finite subgroup $\Gamma\subset G$ (where $G=\mathrm{SL}_2(\mathbb{C})$ as above). This linear connection specifies the initial value (and first derivative) of a solution to $\mathrm{P_{VI}}$. This $\mathrm{P_{VI}}$ solution will have finite monodromy, since we know the branching of $\mathrm{P_{VI}}$ solutions corresponds to the $\mathcal{F}_2$ action on conjugacy classes of triples in G^3, and the orbit through (M_1,M_2,M_3) will be finite, since the action is within triples of generators of Γ.

Thus we see that finite $\mathcal{F}_2$ orbits (in G^3/G) correspond to $\mathrm{P_{VI}}$ solutions with a finite number of branches, and the points of such $\mathcal{F}_2$ orbits correspond to the individual branches of the $\mathrm{P_{VI}}$ solution. In particular the size of the orbit, the number of branches, is the degree of the map $t:\Pi\to\mathbb{P}^1$. (Indeed the $\mathcal{F}_2$ action on such a finite orbit itself gives the full permutation representation of the Belyi map $t:\Pi\to\mathbb{P}^1$, and in particular, by the Riemann–Hurwitz formula, determines the genus of the 'Painlevé curve' Π.)

Said differently it is useful to define a *topological algebraic* P_{VI} *solution* (or henceforth for brevity a *topological solution*) to be a finite $\mathcal{F}_2$ orbit in G^3/G. (The classification of such orbits is still open and is the main step in classifying all finite branching P_{VI} solutions.) In these terms the first paragraph above points out that one obtains 'obvious' topological solutions upon taking any triple of generators of any finite subgroup of G.

For example (omitting discussion of how they were actually constructed), here are some solutions corresponding to certain triples of generators of the binary tetrahedral and octahedral subgroups, due to Dubrovin (1995) and Hitchin (2003) (in different but equivalent forms):

Tetrahedral solution of degree 3

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \quad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)}, \quad \text{and} \quad \theta = (2,1,1,2)/3,$$

Octahedral solution of degree 4

$$y = \frac{(s-1)^2}{s(s-2)}, \quad t = \frac{(s+1)(s-1)^3}{s^3(s-2)}, \quad \text{and} \quad \theta = (1,1,1,1)/4.$$

In both cases Π is a rational curve (with parameter s). Although written in this compact form, one should bear in mind these formulae represent a whole (isomonodromic) family of connections (**A**) as t varies. An explicit elliptic solution appears in Hitchin (1995*a*) and may be written as

Elliptic dihedral solution

$$y = \frac{(3\,s-1)\left(s^2-4\,s-1\right)\left(s^2+u\right)(s\,(s+2)-u)}{(3\,s^3+7\,s^2+s+1)\,(s^2-u)\,(s\,(s-2)+u)},$$

$$t = \frac{\left(s^2+u\right)^2(s\,(s+2)-u)\,(s\,(s-2)-u)}{\left(s^2-u\right)^2(s\,(s+2)+u)\,(s\,(s-2)+u)},$$

where the pair (s,u) lives on the elliptic curve $u^2 = s\left(s^2+s-1\right)$ and $\theta = (1,1,1,1)/2$. This solution has degree 12 and corresponds to a triple of generators of the binary dihedral group of order 20.

It turns out (see Boalch 2006*a*, remark 16) that the icosahedral solutions of Dubrovin and Mazzocco (2000) fit into this framework as well and correspond to (certain) triples of generators of the binary icosahedral group, although in the first instance they arose from the icosahedral reflection group as described earlier. Note that remark 0.1 of Dubrovin and Mazzocco (2000) describes a relation between their solutions of P_{VI} and a certain *folding* of Schwarz's list; this is different to the relation just mentioned – in particular problem (**A**) demands an *extension* of Schwarz's list.

11.4 Beyond Platonic Painlevé VI solutions

My starting point in this project was simply the observation that there should be more algebraic solutions to P_{VI} than those coming from finite subgroups of $SL_2(\mathbb{C})$. Dubrovin (1995) had shown how to relate three-dimensional real orthogonal reflection groups to a certain one-parameter family of the full four-dimensional family of P_{VI} equations (namely, the family having parameters $\theta = (0, 0, 0, *)$) and this was used in Dubrovin and Mazzocco 2000 to classify algebraic solutions having parameters in this one-parameter family. (Some aspects of *op. cit.* were subsequently extended by Mazzocco 2001 to classify rational solutions – i.e. those with only one branch, cf. also Yuan and Li 2002.) The further observation was that if one is able to get away from the orthogonality condition here then one will relate *any* P_{VI} equation to a three-dimensional *complex* reflection group.

Theorem 11.1 (Boalch 2003, 2005) *The isomonodromic deformations of type* (**B**) *connections (on rank 3 vector bundles) are also controlled by the Painlevé VI equation, and all* P_{VI} *equations arise in this way.*

Thus a solution to P_{VI} can also be viewed as specifying an isomonodromic family of rank 3 Fuchsian connections. It turns out that the formulae to go from a P_{VI} solution $y(t)$ to such an isomonodromic family are more symmetrical than in the previous case (type (**A**)) so we will recall them here. (For the analogous formulae for (**A**) see Jimbo and Miwa 1981 and in Harnad's dual picture – the formula for which should be compared to that below – see Harnad 1994 and also Mazzocco 2002, which was kindly pointed out by the referee.) First the parameters: let $\lambda_i = \mathrm{Tr}(B_i)$ for $i = 1, 2, 3$ and let μ_i be the eigenvalues, in some order, of $B_1 + B_2 + B_3$ (which is minus the residue at infinity), so that $\sum \lambda_i = \sum \mu_i$.

Theorem 11.2 (Boalch 2006*b*) *If* $y(t)$ *solves Painlevé VI with parameters* θ *where*

$$\theta_1 = \lambda_1 - \mu_1, \quad \theta_2 = \lambda_2 - \mu_1, \quad \theta_3 = \lambda_3 - \mu_1, \quad \textit{and} \quad \theta_4 = \mu_3 - \mu_2,$$

and we define $x(t)$ *via*

$$x = \frac{1}{2}\left(\frac{(t-1)y' - \theta_1}{y} + \frac{y' - 1 - \theta_2}{y - t} - \frac{t\,y' + \theta_3}{y - 1}\right)$$

then the family of logarithmic connections (**B**) *will be isomonodromic as* t *varies, where*

$$B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \lambda_2 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \quad \textit{and} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \lambda_3 \end{pmatrix}$$

$$b_{12} = \lambda_1 \;\; - \mu_3 y + (\mu_1 - xy)(y-1), \qquad b_{32} = (\mu_2 - \lambda_2 - b_{12})/t,$$
$$b_{13} = \lambda_1 t - \mu_3 y + (\mu_1 - xy)(y-t), \qquad b_{23} = (\mu_2 - \lambda_3)t - b_{13},$$
$$b_{21} = \lambda_2 + \frac{\mu_3(y-t) - \mu_1(y-1) + x(y-t)(y-1)}{t-1}, \quad b_{31} = (\mu_2 - \lambda_1 - b_{21})/t.$$

The implication of this for algebraic solutions should now be clear: the monodromy of a $\mathrm{P_{VI}}$ solution is also described by an action of the free group $\mathcal{F}_2$ on (conjugacy classes of) triples of three-dimensional complex reflections (r_1, r_2, r_3) (with the same formula as before, just replace M_i by r_i). Thus in this context the 'obvious' topological solutions (i.e. finite $\mathcal{F}_2$ orbits) come from taking a triple of generating reflections of a finite complex reflection group in $\mathrm{GL}_3(\mathbb{C})$. Such finite complex reflection groups were classified by Shephard and Todd (1954) and apart from the familiar real orthogonal reflection groups there is an infinite family plus four exceptional complex groups, the Klein reflection group (of order 336, a two-fold cover of Klein's simple group isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7) \hookrightarrow \mathrm{PGL}_3(\mathbb{C})$), two Hessian groups, and the Valentiner group (of order 2,160, a six-fold cover of $A_6 \hookrightarrow \mathrm{PGL}_3(\mathbb{C})$).

The infinite family of groups and the two Hessian groups do not seem to lead to interesting new solutions, but by computing the $\mathcal{F}_2$ orbits (determining the topology of Π) it is easy to see that the Klein group yields a genus 0 degree 7 solution and the Valentiner group has three inequivalent triples of generating reflections, each leading to genus 1 solutions with degrees 15,15, and 24, respectively. These are new solutions, previously undetected. (The 24 appearing here led to a certain amount of trepidation, given that the 10 page elliptic solution of Dubrovin and Mazzocco 2000 had degree 18.)

11.4.1 *Construction*

Of course finding the topological solution is not the same as finding an explicit isomonodromic family of connections; one needs to solve a family of Riemann–Hilbert problems inverting the transcendental Riemann–Hilbert map for each value of t. (Indeed my original plan was to just prove the existence of new interesting solutions, in Boalch (2003), but a certain stubbornness, and some inspiration from reading about Klein's work finding explicit 3×3 matrices generating his simple group, convinced us to go further.)

The two main steps in the method we finally got to work are as follows. (This is a generalization of the method used by Dubrovin and Mazzocco 2000.)

1. Jimbo's asymptotic formulae. Jimbo (1982) found an exact formula for the leading asymptotics at $t = 0$ of the branch of the $\mathrm{P_{VI}}$ solution $y(t)$ corresponding to any sufficiently generic linear monodromy representation (M_1, M_2, M_3). (This formula was obtained by considering the degeneration of the isomonodromic family of connections $(\mathbf{A})$ as $t \to 0$; in the limit the four-punctured sphere degenerates into a stable curve with two components, each

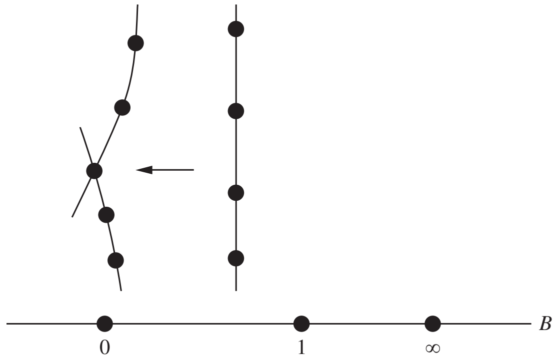

Figure 11.1 Degeneration to two hypergeometric systems.

with three marked points. The connections (**A**) degenerate into hypergeometric systems on each component, with known monodromy (Figure 11.1). Since these are rigid it is easy to solve their Riemann–Hilbert problems explicitly and this gives the leading asymptotics of the isomonodromic family and thus of the P_{VI} solution.)

This is useful for us because, as Jimbo mentions, one may substitute the leading asymptotics back into the P_{VI} equation to get arbitrarily many terms of the precise asymptotic expansion of the solution at 0. If the solution is algebraic, then this is its Puiseux expansion, a sufficient number of terms of which will determine the entire solution.

It turns out there was a typo in Jimbo (1982), which meant the entire method did not work (indeed the fact it did not work led to the questioning of Jimbo's formula and hence the correction in Boalch 2005). (Note the special parameters of Dubrovin and Mazzocco 2000 are not covered by Jimbo's result; rather they adapted the argument of Jimbo 1982 to their case.)

2. Relating (A) and (B). Since Jimbo's formula requires a monodromy representation of a connection of type (**A**), and we are starting with a triple of 3×3 complex reflections (the monodromy representation of a connection of type (**B**)), the second step is that we need to see how to go between these two pictures (on both the De Rham and Betti sides of the Riemann–Hilbert correspondence). This will be described in the following subsection.

11.4.2 *Relating connections* (**A**) *and* (**B**)

We wish to sketch how to convert a connection (**B**) on a rank 3 vector bundle into a connection of the form (**A**) on a rank 2 bundle. On the other side of the Riemann–Hilbert correspondence this amounts to an $\mathcal{F}_2$-equivariant map from triples of complex reflections to triples of elements of $G = \mathrm{SL}_2(\mathbb{C})$ (as in Boalch 2005, section 2).

Of course the monodromy groups change in a highly non-trivial way under this procedure. For example, the Klein reflection group becomes the triangle group $\Delta_{237} \subset G$, which is an infinite group, and the Valentiner group becomes the binary icosahedral group (leading to an unexpected relation between A_6 and A_5).

After this procedure was put on the arxiv (Boalch 2005) we learnt (Dettweiler and Reiter 2007) that it is essentially a case of the middle convolution functor used by Katz (1996), although our construction using the complex analytic Fourier–Laplace transform is different from that of Katz (using l-adic methods) and from the work of Dettweiler and Reiter (2000).

The basic picture which emerges is as follows (see the figure below), and ought to be better known. It was obtained essentially by a careful reading of Balser, Jurkat and Lutz (1981), although the basic idea of relating irregular and Fuchsian systems by the Laplace transform dates back to Birkhoff and Poincaré. (Dubrovin 1995, 1999 used an orthogonal analogue in relation to Frobenius manifolds, also using Balser, Jurkat, and Lutz 1981. Moreover the top triangle is essentially a case of 'Harnad duality' (Harnad 1994) so for $n = 3$ we knew we would obtain all P_{VI} equations.)

$$\begin{tikzcd}
& A \in \mathfrak{gl}_n(\mathbb{C}) \arrow[dl, "A_i = E_iA"] \arrow[dr] & \\
d - \sum \frac{A_i}{z-a_i}dz \arrow[rr, leftrightarrow, "\text{Fourier–Laplace}"'] \arrow[d, "\text{Riemann–Hilbert}"] & & d - \left(\frac{A^0}{z^2} + \frac{A}{z}\right) dz \arrow[d, "\text{Stokes}"'] \\
\begin{array}{c} r_1, \dots, r_n \\ r_i = 1 + e_i \otimes \alpha_i \end{array} \arrow[rr, leftrightarrow, "\text{Killing–Coxeter}", "(\alpha_i(e_j)) = hu_+ - u_-"'] \arrow[dr, "r_n \cdots r_1"'] & & (u_-, u_+, h) \in U_- \times U_+ \times H \arrow[dl, "u_-^{-1} h u_+"] \\
& \mathrm{GL}_n(\mathbb{C}) &
\end{tikzcd}$$

The idea is to describe a transcendental map from $\mathfrak{gl}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{C})$ in two different ways (the two paths down the left and the right from the top to the bottom of the figure).

Choose n distinct complex numbers $a_1, \dots, a_n$ and define $A^0 = \mathrm{diag}(a_1, \dots, a_n)$. Roughly speaking (on a dense open patch) the left-hand column arises by defining $A_i = E_iA$ (setting to zero all but the ith row of A) and constructing the logarithmic connection $d - \sum \frac{A_i}{z-a_i} dz$ having rank 1 residues at each a_i. Then taking the monodromy of this yields n complex reflections r_i (and if bases of solutions are chosen carefully one can naturally define vectors e_i and one-forms α_i such that $r_i = 1 + e_i \otimes \alpha_i$ and that the e_i form a basis). Then the map to $\mathrm{GL}_n(\mathbb{C})$ is given by taking the product of $r_n \cdots r_1$ of these reflections, written in the e_i basis.

Now the key algebraic fact, which dates back at least to Killing (1889) (see Coleman 1989), is that any such product of complex reflection lies in the big cell of $\mathrm{GL}_n(\mathbb{C})$ and so may be factored as the product of a lower triangular and an upper triangular matrix. We write this product as $u_-^{-1} h u_+$ with $u_\pm \in U_\pm$ the unipotent triangular subgroups, and $h \in H$ diagonal:

$$r_n \cdots r_2 r_1 = u_-^{-1} h u_+. \tag{11.2}$$

Further, although this relation between the reflections and $u_\pm$ looks to be highly non-linear, one can relate them in an almost linear fashion: the matrix $hu_+ - u_-$ is the matrix with entries $\alpha_i(e_j)$.

On the other hand it turns out that the same map can be defined by taking the *Stokes data* of the irregular connection $d - \left(\frac{A^0}{z^2} + \frac{A}{z}\right) dz$. Indeed the map on the right-hand side generalizes (Boalch 2002) to any complex reductive group G in place of $\mathrm{GL}_n(\mathbb{C})$, but only for $\mathrm{GL}_n(\mathbb{C})$ is the alternative 'logarithmic' viewpoint available. Thus $u_\pm$ are also the two Stokes matrices of this irregular connection (the natural analogue of monodromy data for such connections); the exact definition is not important here. (The element h is the so-called formal monodromy, explicitly it is simply $\exp(2\pi i \Lambda)$ where Λ is the diagonal part of A.) The two connections are related (see Balser, Jurkat, and Lutz 1981) by the Fourier–Laplace transform: this is more than just formal, and by relating bases of solutions on both sides the stated relation between the Stokes and monodromy data is obtained. (In both cases the resulting element of $\mathrm{GL}_n(\mathbb{C})$ is the monodromy around $z = \infty$ in a suitable basis.) In summary we see that the 'Betti' incarnation of the Fourier–Laplace transform is the relation of Killing–Coxeter.

Now to apply this in the current context we consider the effect of adding a scalar λ to $A \in \mathfrak{gl}_n(\mathbb{C})$. On the right-hand side this corresponds to tensoring the irregular connection by the logarithmic connection $d - \lambda dz/z$ on the trivial line bundle, and Balser, Jurkat, and Lutz (1981) showed that the Stokes data is changed only by scaling h by $s := \exp(2\pi i \lambda)$, fixing $u_\pm$. On the logarithmic side this corresponds to a non-trivial convolution operation, changing the monodromy representation in a non-trivial way. Of course using the Killing–Coxeter identity we now see precisely how the complex reflections vary. (It is perhaps worth noting that this scalar shift is essentially the inverse of the spectral parameter introduced by Killing 1889, p. 20, appearing in the characteristic polynomial of the Killing–Coxeter matrix (11.2): $\det(u_-^{-1} s h u_+ - 1) = \det(s h u_+ - u_-)$.)

If we set $n = 3$ then the logarithmic connections appearing are of the form (**B**), upon taking $a_1, a_2, a_3 = 0, t, 1$. Then we may choose the scalar shift such that the resulting element of $\mathrm{GL}_3(\mathbb{C})$ has 1 as an eigenvalue. This implies that the connections are reducible and we can take the irreducible rank 2 sub- or quotient connection. Projecting to $\mathfrak{sl}_2$ gives the desired connection of type (**A**) (see Boalch 2005). (Note that there is a choice involved here, of which eigenvalue to shift to 1.)

11.4.3 *New solutions*

Thus in summary the procedure now is as follows: take a triple of generating reflections of a finite complex reflection group in $\mathrm{GL}_3(\mathbb{C})$. Push it down to the 2×2 framework using the scalar shift to obtain a triple (M_1, M_2, M_3) of elements of $\mathrm{SL}_2(\mathbb{C})$ in an isomorphic $\mathcal{F}_2$ orbit. Apply Jimbo's formula to get the leading asymptotics of the corresponding P_{VI} solutions at $t = 0$ on each branch (i.e. for each triple in the $\mathcal{F}_2$ orbit). (Converting the values which arise into exact algebraic numbers.) Substitute these leading terms back into P_{VI} to obtain arbitrarily many terms of the Puiseux expansion at 0 of each solution branch. Use these expansions to determine the polynomial $F(y,t)$ defining the solution (assuming it is algebraic). Find a parametrization of the resulting algebraic curve (e.g. using M. van Hoeij's wonderful Maple algebraic curves package).

For example, for the Klein complex reflection group of order 336 this works perfectly (Boalch 2005) and the resulting solution is

Klein solution

$$y = -\frac{\left(5\,s^2-8\,s+5\right)\left(7\,s^2-7\,s+4\right)}{s\,(s-2)\,(s+1)\,(2\,s-1)\,(4\,s^2-7\,s+7)},$$

$$t = \frac{\left(7\,s^2-7\,s+4\right)^2}{s^3\left(4\,s^2-7\,s+7\right)^2}, \quad \text{and } \theta = (2,2,2,4)/7$$

which has 7 branches. One may of course now substitute this back into the formula of Theorem 11.2 (with $\lambda = (1,1,1)/2$ and $\mu = (3,5,13)/14$) to obtain an explicit family of logarithmic connections having monodromy equal to the Klein reflection group generated by reflections (see Boalch 2006*b*, section 3).

When converted to connections of type (**A**) these 'Klein connections' have infinite (projective) monodromy group equal to the triangle group Δ_{237} (cf. Boalch 2006*c*, appendix B). On the other hand it turns out (Boalch 2006*a*) that for the Valentiner connections, even though they are much trickier to construct directly, we can still compute immediately that they become connections of type (**A**) with binary icosahedral monodromy. They are also inequivalent to those appearing in the work of Dubrovin and Mazzocco related to the real orthogonal icosahedral reflection group (which lead to unipotently generated monodromy with one choice of the scalar shift, but finite binary icosahedral monodromy with a different choice, cf. Boalch 2006*a*, remark 16).

Thus it seemed like a good idea to examine precisely what P_{VI} solutions arise upon taking arbitrary triples of generators (M_1, M_2, M_3) of the binary

Table 11.2. Icosahedral solutions 11–52

	Degree	Genus	Walls	Type		Degree	Genus	Walls	Type
11	2	0	2	$b^2\,c^2$	32	10	0	3	d^4
12	2	0	2	$b^2\,d^2$	33	12	0	0	$a\,b\,c\,d$
13	2	0	2	$c^2\,d^2$	34	12	1	1	$a\,b\,c^2$
14	3	0	1	$b\,c^2\,d$	35	12	1	1	$a\,b\,d^2$
15	3	0	1	$b\,c\,d^2$	36	12	1	1	$b^2\,c\,d$
16	4	0	2	$a\,c^3$	37	15	1	2	$b^3\,c$
17	4	0	2	$a\,d^3$	38	15	1	2	$b^3\,d$
18	4	0	2	$c^3\,d$	39	15	1	2	$b^2\,c^2$
19	4	0	2	$c\,d^3$	40	15	1	2	$b^2\,d^2$
20	5	0	1	$b^2\,c\,d$	41	18	1	3	b^4
21	5	0	2	$c^2\,d^2$	42	20	1	1	$a\,b^2\,c$
22	6	0	1	$b\,c^2\,d$	43	20	1	1	$a\,b^2\,d$
23	6	0	1	$b\,c\,d^2$	44	20	1	3	$a^2\,c^2$
24	8	0	1	$a\,c^2\,d$	45	20	1	3	$a^2\,d^2$
25	8	0	1	$a\,c\,d^2$	46	24	1	2	$a\,b^3$
26	9	1	2	$b\,c^3$	47	30	2	2	$a^2\,b\,c$
27	9	1	2	$b\,d^3$	48	30	2	2	$a^2\,b\,d$
28	10	0	2	$a^2\,c\,d$	49	36	3	3	$a^2\,b^2$
29	10	0	2	$b^3\,c$	50	40	3	3	$a^3\,c$
30	10	0	2	$b^3\,d$	51	40	3	3	$a^3\,d$
31	10	0	3	c^4	52	72	7	3	$a^3\,b$

icosahedral group. Thus we looked at all triples of generators and quotiented by the relation coming from the affine F_4 symmetries of $\mathrm{P_{VI}}$. The resulting table has 52 rows (which is quite small considering there are 26,688 conjugacy classes of generating triples). The first 10 rows correspond to the 10 icosahedral rows of Schwarz's list and thus the projective monodromy around one of the four punctures is the identity (these correspond to the $\mathrm{P_{VI}}$ solution $y = t$). The remaining rows are as in Table 11.2 (this is abridged from Boalch 2006*a*). (Note that the right notion of equivalence in the linear non-rigid problem (**A**) seems to be the 'geometric equivalence' of Boalch 2006*a*, section 4 – however this coincides with equivalence under the affine F_4 Weyl group, in this case.)

Thus there are lots of other icosahedral solutions the largest having genus 7 and 72 branches. (The column 'Type' indicates the set of conjugacy classes of local monodromy of the corresponding connections of type (**A**), as we marked on Schwarz's list. The column 'Walls' indicates the number of reflection hyperplanes for the affine F_4 Weyl group that the solution's parameters θ lie on.) A few of these solutions had appeared before: those with degree <5 are simple deformations of previous solutions, solutions 21 and 26 are in Kitaev (2005) and the Dubrovin–Mazzocco icosahedral solutions are equivalent to those on rows 31,32, and 41. On the other hand the Valentiner solutions are quite far down the list on rows $37, 38$, and 46.

The above method of constructing solutions using Jimbo's asymptotic formula applies only to sufficiently generic monodromy representations but it turns out that most of the rows of this table have some representative (in their affine F_4 orbit) to which Jimbo's formula maybe applied (on every branch). Thus we could start working down the list constructing new solutions. An initial goal was to get to solution 33: this solution purports to be on *none* of the reflection hyperplanes and the folklore was that all explicit solutions to Painlevé equations must lie on some reflection hyperplane. The folklore was wrong:

'Generic' solution/Icosahedral solution 33

$$y = -\frac{9s(s^2+1)(3s-4)(15s^4-5s^3+3s^2-3s+2)}{(2s-1)^2(9s^2+4)(9s^2+3s+10)},$$

$$t = \frac{27s^5(s^2+1)^2(3s-4)^3}{4(2s-1)^3(9s^2+4)^2}, \quad \text{and}$$

$$\theta = (2/5, 1/2, 1/3, 4/5).$$

So far this looks to be the only example of a 'classical' solution of any of the Painlevé equations that does not lie on a reflection hyperplane (of the full symmetry group). Apart from being in the interior of a Weyl alcove this solution is generic in another sense: a randomly chosen triple of generators of the binary icosahedral group is most likely to lead to it (more of the 26,688 triples of generators correspond to this row than to any other). Notice also that this solution has type *abcd*; there is one local monodromy in each of the four non-trivial conjugacy classes of A_5.

At this stage we were approaching solution 41 which we knew took 10 pages to write down. So we stopped and looked around to see if there were other interesting (even just topological) solutions. (The tetrahedral and octahedral cases could all now be fully dealt with Boalch 2006*c*.)

11.5 Pullbacks

In his 1884 book on the icosahedron (see Klein 1956), Klein showed that all second-order Fuchsian differential equations with finite monodromy are (essentially) pullbacks of a hypergeometric equation along a rational map f:

$$\begin{tikzcd}
\mathbb{P}^1\setminus\{k \text{ points}\} \arrow[rr, dashed, "\text{Schwarz map}", "y_1/y_2"'] \arrow[dr, "f"'] & & \mathbb{P}^1 \arrow[dl, "/\Gamma\ (\text{invariants})"] \\
& \mathbb{P}^1\setminus\{0,1,\infty\} &
\end{tikzcd}$$

In particular ($k = 3$) all the icosahedral entries on Schwarz's list may be obtained by pulling back the '235' hypergeometric equation (on row VI of Schwarz's list).

In our context, an isomonodromic family of connections of type (**A**) amounts to a family of Fuchsian equations with five singularities (at $0, t, 1, \infty$, plus an apparent singularity at another point y).[2] Klein's theorem says each element of this family arises as the pullback of the 235 hypergeometric equation along a rational map, so the family corresponds to a family of rational maps.

Thus finding a P_{VI} solution corresponding to a family of connections (**A**) with finite monodromy amounts to giving a certain family of rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$. To construct such P_{VI} solutions one may try to find such families of rational maps, such that each map pulls back a hypergeometric equation to an equation with the right number of singular points – or to one that can be put in this form after using elementary transformations to remove extraneous apparent singularities. (This is not straightforward; e.g. given a finite monodromy representation of a connection (**A**) it is not immediate even what degree such a map f will have.)

An important further observation (due to C. Doran 2001 and A. Kitaev 2002) is that any such family of rational maps will lead to algebraic solutions of Painlevé VI regardless of whether or not the hypergeometric equation being pulled back has finite monodromy (provided the equation upstairs has the right number of poles); the algebraicity follows from that of the family of rational maps.

Andreev and Kitaev (2002); Kitaev (2002, 2005) have used this to construct some P_{VI} solutions, essentially by starting to enumerate all such rational maps (this leads to a few new solutions, but most in fact turn out to be equivalent to each other or to ones previously constructed – see Section 11.7).

On the other hand, Doran had the idea that interesting P_{VI} solutions should come from hypergeometric equations with interesting monodromy groups. Thus (amongst other things) Doran (2001) studied the possible hypergeometric equations with monodromy a *hyperbolic arithmetic triangle group* which may be pulled back to yield P_{VI} solutions. Indeed in Doran (2001, corollary 4.6), he lists such possible triangle groups and the degrees and ramification indices of the corresponding rational maps f, although no new solutions were actually constructed. We picked up on this thread in Boalch (2006*c*, section 5): it was found that all but one entry on Doran's list corresponded to a known explicit solution (although were perhaps unknown when Doran 2001 was published). The remaining entry was for a family of degree 10 rational maps f pulling back the 237 triangle group with ramification indices (partitions of 10):

$$[2, 2, 2, 2, 2],\ [3, 3, 3, 1],\ [7, 1, 1, 1]$$

over $0, 1, \infty$ (where the hypergeometric system has projective monodromy of orders 2, 3, and 7, respectively), as well as minimal ramification $[1^8, 2]$ over another variable point. As explained in Boalch (2006*c*) one can get from here to

[2] This is the same y appearing in P_{VI} – that is, the function y on the space of connections (**A**) is the position of the apparent singularity that appears when the connection is converted into a Fuchsian equation (Fuchs 1905).

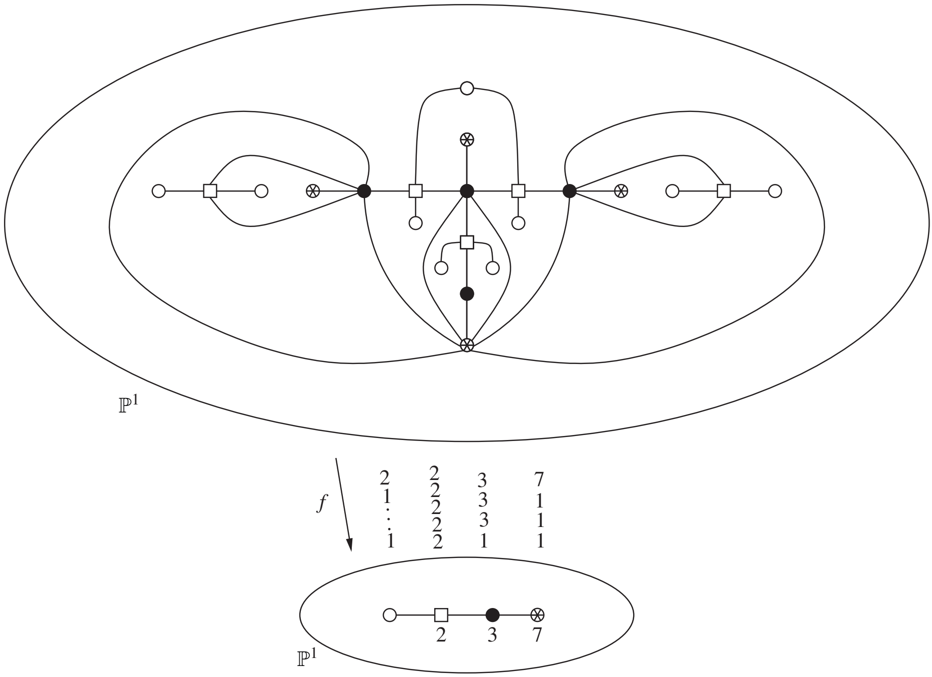

Figure 11.2 237 degree 10 rational map f.

a topological P_{VI} solution by drawing a picture: we wish to find such a rational map f topologically – that is, describe the topology of a branched cover $f: \mathbb{P}^1 \to \mathbb{P}^1$ with this ramification data. This may be done by playing 'join the dots' (completely in the spirit of Grothendieck's Dessins d'Enfants) and yields a covering figure as required. One figure so obtained is shown in Figure 11.2. (Note that, in the context of Painlevé equations, the idea of drawing pictures such as Figure 11.2 first appeared in Kitaev (2005).)

The upper copy of $\mathbb{P}^1$ is thus divided into 10 connected components and f maps each component isomorphically onto the complement of the interval drawn on the lower $\mathbb{P}^1$ (the lines and the vertices upstairs are the preimages of the lines and vertices downstairs). In particular the figure shows how loops upstairs map to words in the generators of the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ downstairs. In this way we can compute by hand the monodromy of the equation upstairs obtained by pulling back a hypergeometric equation with monodromy Δ_{237}. This yields the triple:

$$M_1 = caca^{-1}c^{-1}, \quad M_2 = c, \quad \text{and} \quad M_3 = c^{-1}a^{-1}cac$$

(where $a, b,$ and c are lifts to $\mathrm{SL}_2(\mathbb{C})$ of standard generators of Δ_{237} with $cba = 1$), which we know a priori lives in a finite $\mathcal{F}_2$ orbit. One finds immediately that the orbit through the conjugacy class of this triple has size 18 and constitutes a genus 1, degree 18 topological $\mathrm{P_{VI}}$ solution.

Now it turns out that Jimbo's formula may be applied to every branch of this solution, and proceeding as before we obtain (Boalch 2006*c*) the solution explicitly:

Elliptic 237 solution

$$y=\frac{1}{2}-\frac{\left(3\,s^8-2\,s^7-4\,s^6-204\,s^5-536\,s^4-1738\,s^3-5064\,s^2-4808\,s-3199\right)u}{4\,(s^6+196\,s^3+189\,s^2+756\,s+154)\,(s^2+s+7)\,(s+1)},$$

$$t=\frac{1}{2}-\frac{\left(s^9-84\,s^6-378\,s^5-1512\,s^4-5208\,s^3-7236\,s^2-8127\,s-784\right)u}{432\,s\,(s+1)^2\,(s^2+s+7)^2},$$

where $u^2=s\,(s^2+s+7)$ and $\theta=(2/7,2/7,2/7,1/3)$. (This solution, or rather an inequivalent 'Galois conjugate' of it, has also been obtained independently by Kitaev 2006, p. 219 by directly computing such a family of rational maps – apparently also influenced by Doran's list.)

11.6 Final steps

11.6.1 *Up to degree 24*

We now have an example of a degree 18 elliptic solution to Painlevé VI with a quite simple form. This leads immediately to the suspicion that the 10-page Dubrovin–Mazzocco solution is just written at a bad value of the parameters. Indeed using the method we have been 'tweaking' while working down the icosahedral table enables us to guess good a priori choices of the parameters θ within the corresponding affine F_4 equivalence class (row 41 in Table 11.2) i.e. so that the expression for the polynomial F will be 'small'. Choosing such parameters and constructing the solution from scratch at those parameters yields

Theorem 11.3 (Boalch 2006*a*) *The Dubrovin–Mazzocco icosahedral solution is equivalent to the solution*

$$y=\frac{1}{2}-\frac{8\,s^7-28\,s^6+75\,s^5+31\,s^4-269\,s^3+318\,s^2-166\,s+56}{18\,u\,(s-1)\,(3\,s^3-4\,s^2+4\,s+2)}\quad\text{and}$$

$$t=\frac{1}{2}+\frac{(s+1)\left(32\,(s^8+1)-320\,(s^7+s)+1112\,(s^6+s^2)-2420\,(s^5+s^3)+3167\,s^4\right)}{54\,u^3\,s\,(s-1)}$$

on the elliptic curve

$$u^2=s\,(8\,s^2-11\,s+8)$$

with $\theta=(1,1,1,1)/3$. *In particular this elliptic curve is birational to that defined by the 10-page polynomial.*

Substituting this into the formula of Theorem 11.2 with $\lambda = (1,1,1)/2$ and $\mu = (1,3,5)/6$ now gives explicitly the third (and trickiest) family of connections of type (**B**) with monodromy the icosahedral reflection group.

This can be pushed further with more tweaking to get up to degree 24 (row 46 in Table 11.2), that is, to obtain the largest Valentiner solution (Boalch 2006*a*) (the main further tricks used are described in (Boalch 2006*c*, appendix C). In particular this finishes the construction of all *elliptic* icosahedral solutions. Intriguingly, one finds that the resulting elliptic icosahedral Painlevé curves Π become singular only on reduction modulo the primes 2, 3, and 5 (except for rows 44 and 45 – we will see another reason in the following subsection that these are abnormal). Similarly the elliptic Painlevé curve related to the 237 triangle group becomes singular only on reduction modulo 2, 3, and 7.

11.6.2 *Quadratic/Landen/folding transformations*

Now the happy fact is that the remaining icosahedral solutions may be obtained from earlier solutions by a trick, first introduced in the context of P_{VI} by Kitaev (1991) and a simpler equivalent form was found by Ramani, Grammaticos, and Tamizhmani (2000). Manin (1998) refers to some equivalent transformations as Landen transformations. (Landen has clear precedence since the original Landen transformations were rediscovered by Gauss!) Tsuda, Okamoto, and Sakai (2005) call them folding transformations.

In any case the basic idea is simple: if one has a connection (**A**) with two local projective monodromies of order 2 (say at $0, \infty$) then one can pull it back along the map $z \mapsto z^2$ and obtain a connection with only apparent singularities at $0, \infty$ (which can be removed) and four genuine singularities. This can be normalized into the form (**A**), and the key point is that this works in families and maps isomonodromic deformations of the original connections to isomonodromic deformations of the resulting connections – that is, it transforms certain solutions of P_{VI} into different, generally inequivalent, solutions. Of course this is not a genuine symmetry of P_{VI} since special parameters are required, but it is precisely what is needed to construct the remaining solutions.

Indeed observe that each of the rows of the icosahedral table with degree >24 have type $a^2\xi\eta$ for some $\xi, \eta \in \{a, b, c, d\}$ – that is, they have two projective monodromies of order 2. Pulling back along the squaring map will transform the corresponding connections into connections of type $\xi^2\eta^2$. It turns out (in this icosahedral case) the corresponding P_{VI} solutions have half the degree, and we obtain an algebraic relation between the solutions. This program is carried out in Boalch (2007) and the remaining icosahedral solutions are obtained (see also Kitaev and Vidūnas 2007). (Notice also that the elliptic solutions on rows 44 and 45 are related in this way to earlier, genus zero solutions.) For example, in Boalch (2007) we found an explicit equation for the genus 7 algebraic curve naturally attached to the icosahedron, on which the largest (degree 72) icosahedral solution is defined: it may be modelled as the plane octic with affine equation

Genus 7 icosahedral Painlevé curve

$$9\,(p^6\,q^2 + p^2\,q^6) + 18\,p^4\,q^4 +$$
$$4\,(p^6 + q^6) + 26\,(p^4\,q^2 + p^2\,q^4) + 8\,(p^4 + q^4) + 57\,p^2\,q^2 +$$
$$20\,(p^2 + q^2) + 16 = 0.$$

11.7 Conclusion

Thus in conclusion we have filled in a number of rows of what could be called the *non-linear Schwarz's list*. Whether or not there will be other rows remains to be seen. So far this list of known algebraic solutions to $\mathrm{P_{VI}}$ takes the following shape (we will use the letters d and g to denote the degree and genus of solutions, and consider solutions up to equivalence under Okamoto's affine F_4 symmetry group. Some non-trivial work has been done to establish which of the published solutions are equivalent to each other and which were genuinely new). See also Boalch (2006d).

First there are the **rational solutions** ($d = 1$), studied by Mazzocco (2001) and Yuan and Li (2002), which fit into the set of Riccati solutions classified by Watanabe (1998). (Beware that 'rational' here means the solution is a rational function of t, which implies, but is by no means equivalent to, having a rational parameterization.)

Then there are **three continuous families** of solutions $g = 0, d = 2, 3, 4$. The degree 2 family is $y = \sqrt{t}$ which, as one may readily verify, solves $\mathrm{P_{VI}}$ for a family of possible parameter values. Similarly the degree 3 tetrahedral solution, and the degree 4 octahedral and dihedral solutions (of Dubrovin 1995 and Hitchin 1995a, 2003) fit into such families, as discussed in Ben Hamed and Gavrilov (2005); Boalch (2006a) and Cantat and Loray (2007). In general in such a family $y(t)$ may depend on the parameters of the family. Ben Hamed and Gavrilov (2005) showed that any family with $y(t)$ *not* depending on the parameters is equivalent to one of the above cases and recently Cantat and Loray (2007) showed that any solution with two, three, or four branches is in such family.

Next there is **one discrete family** (d, g unbounded, $\theta = (0, 0, 0, 1) \sim (1, 1, 1, 1)/2$). Indeed this $\mathrm{P_{VI}}$ equation was solved completely by Picard (1889, p. 299), Fuchs (1905), and in a different way by Hitchin (1995b). Algebraic (determinantal) formulae for the algebraic solutions amongst these appear in Hitchin (1995a), using links with the Poncelet problem – in this framework they are dihedral solutions (controlling connections of type (**A**) with binary dihedral monodromy).

Finally there are **45 exceptional solutions**, which collapse down to **30** if we identify solutions related by quadratic transformations. The possible genera are $0, 1, 2, 3, 7$, and the highest degree is 72. Of these 30 solutions 7 have previously appeared: 1 is due to Dubrovin (1995), 2 to Dubrovin and Mazzocco (2000), and 4 to Kitaev (3 in Kitaev 2005, plus – in Kitaev 2006 – a Galois conjugate of the elliptic 237 solution already mentioned). Two of these exceptional solutions are octahedral, 1 is the Klein solution, 3 are the elliptic 237 solution (and its 2 Galois conjugates), and the remaining 24 are icosahedral.

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